

Perfect A/D conversion of entanglement

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Abstract

We investigate how entanglement can be perfectly transfered between continuous variable and qubits system. We find that a two-mode squeezed vacuum state can be converted to the product state of an infinite number of two-qubit states while keeping the entanglement. The reverse process is also possible. The interaction Hamiltonian is a kind of non-linear Jaynes-Cummings Hamiltonian.

PACS: 03.67.Mn; 03.65.Ud

Keywords: perfect entanglement conversion; non-linear Jaynes-Cummings model.

Quantum information processing (QIP) has been extensively studied for a qubit system which is a quantum extension of a bit, spanning two-dimensional Hilbert space. A qubit is realized by a electronic spin, a two-level atom, the polarization of a photon and a superconductor among others. Parallely, much attentions have been paid to the QIP of quantum continuous variable (CV) system which is a quantum extension of analog information in classical information theory. CV physical systems such as a harmonic oscillator, a rotator and a light field are defined in infinite-dimensional Hilbert space. While conversions of analog to digital (A/D) and digital to analog (D/A) are quite usual in information processing, qubit and CV systems are nearly always treated separately. There have been some pilot works on how to entangle two separate qubits by an entangled Gaussian field, the efficient of the transfer is not high. We would propose a scheme of perfect transferring the entanglement in this letter.

The two-mode squeezed vacuum state $|\Psi\rangle_{AB} = \sqrt{1-\lambda^2} \sum_{m=0}^{\infty} \lambda^m |m\rangle_A \otimes |m\rangle_B$, where $\lambda = \tanh r$ with r the squeezing parameter. The entanglement of the state is $E(|\Psi\rangle) = \cosh^2 r \log \cosh^2 r - \sinh^2 r \log \sinh^2 r$. The interaction between different systems can cause the transfer of entanglement between the systems. The scheme of the system considered is that two individual qubits each interacting with one entangled part of the field. The whole system will evolve in the way of $U(t) \rho_{AB}(0) \otimes \rho_{CD}(0) U^\dagger(t)$, where $U(t) = \exp[-\frac{i}{\hbar}(H_{AC} + H_{BD})t]$ is the evolution operator in interaction picture, and $\rho_{AB}(0) = |\Psi\rangle_{AB} \langle\Psi|$ is the initial state of the CV system while $\rho_{CD}(0) = |-\rangle_C \langle-| \otimes |-\rangle_D \langle-|$ is the initial state of the qubit system. Firstly suppose the model Hamiltonian of entanglement transfer from CV system to qubit system or vice versa is

$$H_1 = \hbar\Omega (\sqrt{n}a^+ \sigma_- + a\sqrt{n}\sigma_+), \quad (1)$$

where a and a^+ are the photon annihilation and generation operators respectively, $n = a^+a$, σ_- and σ_+ are operators which convert the atom from its excited state $|+\rangle$ to ground state $|-\rangle$ and from ground state to excited state respectively. The Hamiltonian (1) can be considered as a kind of nonlinear Jaynes-Cummings model[?]. Then $\exp[-\frac{i}{\hbar}H_1 t_1] |m, -\rangle = \cos(m\Omega t_1) |m, -\rangle - i \sin(m\Omega t_1) |m-1, +\rangle$. If the interaction time t_1 is adjusted in such a way that $\Omega t_1 = \pi/2$ then $\exp[-\frac{i}{\hbar}H_1 t_1] |2m, -\rangle = (-1)^m |2m, -\rangle$ and $\exp[-\frac{i}{\hbar}H_1 t_1] |2m+1, -\rangle = -i(-1)^m |2m, +\rangle$. Apply the evolution operator $U_1(t_1) = \exp[-\frac{i}{\hbar}(H_{1AC} + H_{1BD})t_1]$ to the state $|\Psi\rangle_{AB} |--\rangle_{CD}^{(1)}$, then

$$U_1(t_1) |\Psi\rangle_{AB} |--\rangle_{CD}^{(1)} = |\Psi\rangle_{AB}^{(1)} |\Phi\rangle_{CD}^{(1)} \quad (2)$$

with $|\Psi\rangle_{AB}^{(1)} = \sqrt{1-\lambda^4} \sum_{m=0}^{\infty} \lambda^{2m} |2m\rangle_A |2m\rangle_B$ and $|\Phi\rangle_{CD}^{(1)} = \frac{1}{\sqrt{1+\lambda^2}} (|--\rangle_{CD}^{(1)} - \lambda |++\rangle_{CD}^{(1)})$. It should be noticed that the state after evolution is a product state of CV system state and two qubit state. The CV state $|\Psi\rangle_{AB}^{(1)}$ has even number of photons in each mode. We can separate the two qubit state $|\Phi\rangle_{CD}^{(1)}$ from the combined state, then append another vacuum two qubit state $|--\rangle_{CD}^{(2)}$ of CD partite to state $|\Psi\rangle_{AB}^{(1)}$, the new state will be $|\Psi\rangle_{AB}^{(1)} |--\rangle_{CD}^{(2)}$. We would design another interaction Hamiltonian to assign the entanglement

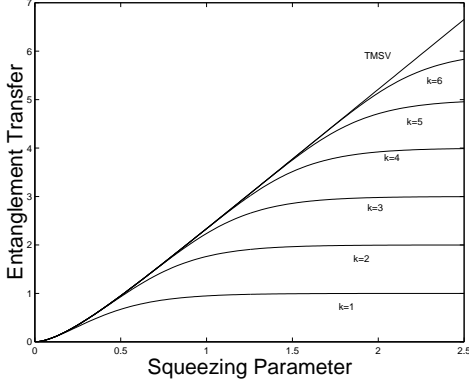


Figure 1: .

of CV state to two qubit state. The Hamiltonian will be $H_2 = \hbar\Omega \left(\sqrt{n}a^+ \frac{1}{\sqrt{n}}a^+ \sigma_- + a \frac{1}{\sqrt{n}}a \sqrt{n}\sigma_+ \right)$, the evolution will be $U_2(t_2) |\Psi\rangle_{AB}^{(1)} |--\rangle_{CD}^{(2)} = |\Psi\rangle_{AB}^{(2)} |\Phi\rangle_{CD}^{(2)}$ with the interaction time $t_2 = \pi/(4\Omega)$, and $|\Psi\rangle_{AB}^{(2)} = \sqrt{1-\lambda^8} \sum_{m=0}^{\infty} \lambda^{4m} |4m\rangle_A |4m\rangle_B$, $|\Phi\rangle_{CD}^{(2)} = \frac{1}{\sqrt{1+\lambda^4}} (|--\rangle_{CD}^{(2)} - \lambda^2 |++\rangle_{CD}^{(2)})$. Then we move from the second two qubit to the vacuum state of the third two qubit of CD partite and so on. The k -th Hamiltonian will be $H_k = \hbar\Omega [n(\frac{1}{\sqrt{n}}a^+)^{2^{k-1}} \sigma_- + (a \frac{1}{\sqrt{n}})^{2^{k-1}} n \sigma_+]$ and interaction time $t_k = \pi/(2^k\Omega)$. The whole state will be

$$U_k(t_k) \cdots U_2(t_2) U_1(t_1) |\Psi\rangle_{AB} (|--\rangle^{(1)}_{CD} |--\rangle^{(2)}_{CD} \cdots |--\rangle^{(k)}_{CD}) = |\Psi\rangle_{AB}^{(k)} (|\Phi\rangle^{(1)}_{CD} |\Phi\rangle^{(2)}_{CD} \cdots |\Phi\rangle^{(k)}_{CD}), \quad (3)$$

with $|\Psi\rangle_{AB}^{(k)} = \sqrt{1-\lambda^{2^{k+1}}} \sum_{m=0}^{\infty} \lambda^{2^k m} |2^k m\rangle_A |2^k m\rangle_B$, $|\Phi\rangle_{CD}^{(k)} = \frac{1}{\sqrt{1+\lambda^{2^k}}} (|--\rangle_{CD}^{(k)} - \lambda^{2^{k-1}} |++\rangle_{CD}^{(k)})$. The entanglement transferred to qubits system is

$$\begin{aligned} E\left(\prod_{j=1}^k |\Phi\rangle_{CD}^{(j)}\right) &= \sum_{j=1}^k E(|\Phi\rangle_{CD}^{(j)}) = \sum_{j=1}^k [\log(1+\lambda^{2^j}) - \frac{\lambda^{2^j}}{1+\lambda^{2^j}} 2^j \log \lambda] \\ &= \log \frac{1-\lambda^{2^{k+1}}}{1-\lambda^2} - \left(\frac{\lambda^2}{1-\lambda^2} - \frac{2^k \lambda^{2^{k+1}}}{1-\lambda^{2^{k+1}}} \right) \log \lambda^2. \end{aligned} \quad (4)$$

The entanglement remained at the CV system is $E(|\Psi\rangle_{AB}^{(k)}) = -\log(1-\lambda^{2^{k+1}}) - \frac{2^{k+1}\lambda^{2^{k+1}}}{1-\lambda^{2^{k+1}}} \log \lambda$. The total entanglement remains unchanged for each k , $E(\prod_{j=1}^k |\Phi\rangle_{CD}^{(j)}) + E(|\Psi\rangle_{AB}^{(k)}) = E(|\Psi\rangle_{AB})$. When $k \rightarrow \infty$, $\lambda^{2^{k+1}} \rightarrow 0$, thus $E(|\Psi\rangle_{AB}^{(k)}) \rightarrow 0$, the entanglement transferred to the qubit system tends to $E(|\Psi\rangle_{AB})$. The entanglement is perfectly transferred. The entanglement transfer is depicted in Fig. (1) for different value of receiving qubit pair number k .

Conversion of a digital number (a serial of bits) to an analog quantity has the property that each bit is independent of other bits. No correlations among these bits exist. That is the source is a discrete memoryless source. D/A conversion is simply convert binary number to M-nary number. The extension of the independence to quantum situation is that there are no entanglements among the series of qubits. In the bipartite case, the state before conversion will be a direct product of a series of two qubits. We have the initial state $|\phi_1\rangle_{CD} |\phi_2\rangle_{CD} \cdots |\phi_k\rangle_{CD}$, where $|\phi_i\rangle_{CD} = a_{00}^i |--\rangle^i + a_{01}^i | -+\rangle^i + a_{10}^i | +-\rangle^i + a_{11}^i | ++\rangle^i$. The process of entanglement transfer is to transfer firstly the higher qubit pair (k -th) to the CV bipartite state then the lower. The result of conversion will be $U_1^+(t_1) |\phi_1\rangle_{CD} U_2^+(t_2) |\phi_2\rangle_{CD} \cdots U_k^+(t_k) |\phi_k\rangle_{CD} |00\rangle_{AB} = |\psi\rangle_{AB} \prod_{i=1}^k |--\rangle^i$, where $|\psi\rangle_{AB} = \sum_{n_1, \dots, n_k, m_1, \dots, m_k=0}^1 \prod_{j=1}^k (-1)^{m_{j+1}+n_{j+1}} i^{m_j+n_j} a_{n_j m_j}^j |n_k \cdots n_1, m_k \cdots m_1\rangle$, with $n = \sum_{j=1}^k n_j 2^{j-1}$ denoted as $n_k \cdots n_1$, $n_j = 0, 1$. The Entanglement of the state $|\psi\rangle_{AB}$ is equal to that of a state $|\psi'\rangle = \sum_{n_1, m_1=0}^1 i^{m_1+n_1} a_{n_1 m_1}^1 |n_1, m_1\rangle \prod_{j=2}^k (\sum_{n_j, m_j=0}^1 (-i)^{m_j+n_j} a_{n_j m_j}^j |n_j, m_j\rangle)$, thus it is equal to the sum of entanglements of qubit pairs. We have $E(|\psi\rangle_{AB}) = \sum_{j=1}^k E(|\phi_j\rangle_{CD})$. The conversion procedure will convert a

general qubit pair product state $\rho_{1CD} \otimes \rho_{2CD} \otimes \cdots \otimes \rho_{kCD}$ into a continuous variable state ρ_{AB} while keeping the entanglement due to local unitary operations.

Conversion of an analog quantity to a digital number has quantization error due to the finite number of the destination bits. Speech signal is converted to eight bits after sampling according to the standard protocol. The quantization error is small enough that it can not be sensed by ear. The quantization of a quantum CV will also has quantization error if a finite number of qubits is used. Some prior works have been done with the quantum rate distortion theory[?] [?]. Let us consider quantum A/D conversion of a single mode quantum state first. The initial CV state is $\rho = \sum_{n,m=0}^{\infty} c_{nm} |n\rangle \langle m|$, the first step of conversion will be $\exp[-\frac{i}{\hbar} H_1 t_1] \rho \otimes |-\rangle \langle -| \exp[\frac{i}{\hbar} H_1 t_1] = \sum_{n,m=0}^{\infty} (-1)^{m+n} |2n\rangle \langle 2m| (c_{2n,2m} |-\rangle \langle -| + i c_{2n,2m+1} |-\rangle \langle +| - i c_{2n+1,2m} |+\rangle \langle -| + c_{2n+1,2m+1} |+\rangle \langle +|)$. Then the unitary transformation $\exp[-\frac{i}{\hbar} H_2 t_2]$ is applied and so on, at last $\exp[-\frac{i}{\hbar} H_k t_k]$ is applied. Each item of the CV part will convert to a form of $|2^k n\rangle \langle 2^k m|$. At this stage the entropy of the state remains intact. We obtain the qubit series by tracing out the CV part and drop it. The tracing operation will increase the total entropy by the triangle relation of the entropies. The result k -qubit state usually has correlation among the qubits. The correlation may even be considered as a kind of entanglement. Let us consider the conversion of a CV state to two qubits, the result state after tracing CV part will be $\sum_{k1,k2,l1,l2=0}^1 i^{k1-k2-l1+l2} d_{k1k2l1l2} |k2, k1\rangle \langle l2, l1|$. Where $|-\rangle, |+\rangle$ are re-expressed by $|0\rangle, |1\rangle$, $d_{k1k2l1l2} = \sum_m c_{4m+2k1+k2, 4m+2l1+l2}$. There may be entanglement between the first and second qubits. For example, when the initial CV state is a coherent state $|\alpha\rangle$ with real parameter α , the entanglement can be quite high. The concurrence increases from 0 ($\alpha = 0$) to 0.9462 ($\alpha = 1.29$), then decreases to 0.8271 ($\alpha = 1.92$), after that it monotonically increases and at $\alpha = 3.4$ revives to the first maximum. Thus quantum A/D conversion can produce entanglement with quite high quality. The entanglement produced from coherent state relies on the phase angle of the complex parameter, for a coherent state with pure imaginary parameter, there is no converted entanglement at all between the two qubits.

An example of totally memoryless qubits series can be obtained from converting the thermal state. The conversion fidelity can be considered at the basis of CV and qubit series as well. We can convert the k -qubit series back to CV state and we obtain a state ρ' , the conversion fidelity F is the fidelity between ρ and ρ' . The distortion is $D = 1 - F$. The rate distortion problem is that what is the minimal coding rate R for a given source under the constrain of distortion D . Thermal state source $\rho = (1-v) \sum_{n=0}^{\infty} v^n |n\rangle \langle n|$ is one of the simplest source. $\rho' = \frac{(1-v)}{1-v^{2^k}} \sum_{n=0}^{2^k-1} v^n |n\rangle \langle n|$. The distortion will be $D = 1 - \sqrt{1 - v^{2^k}}$, Thus $k = \log[\log[1 - (1 - D)^2] / \log v]$.

References

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